$$1 + 2 + 3 + \cdots = -\frac{1}{12}$$

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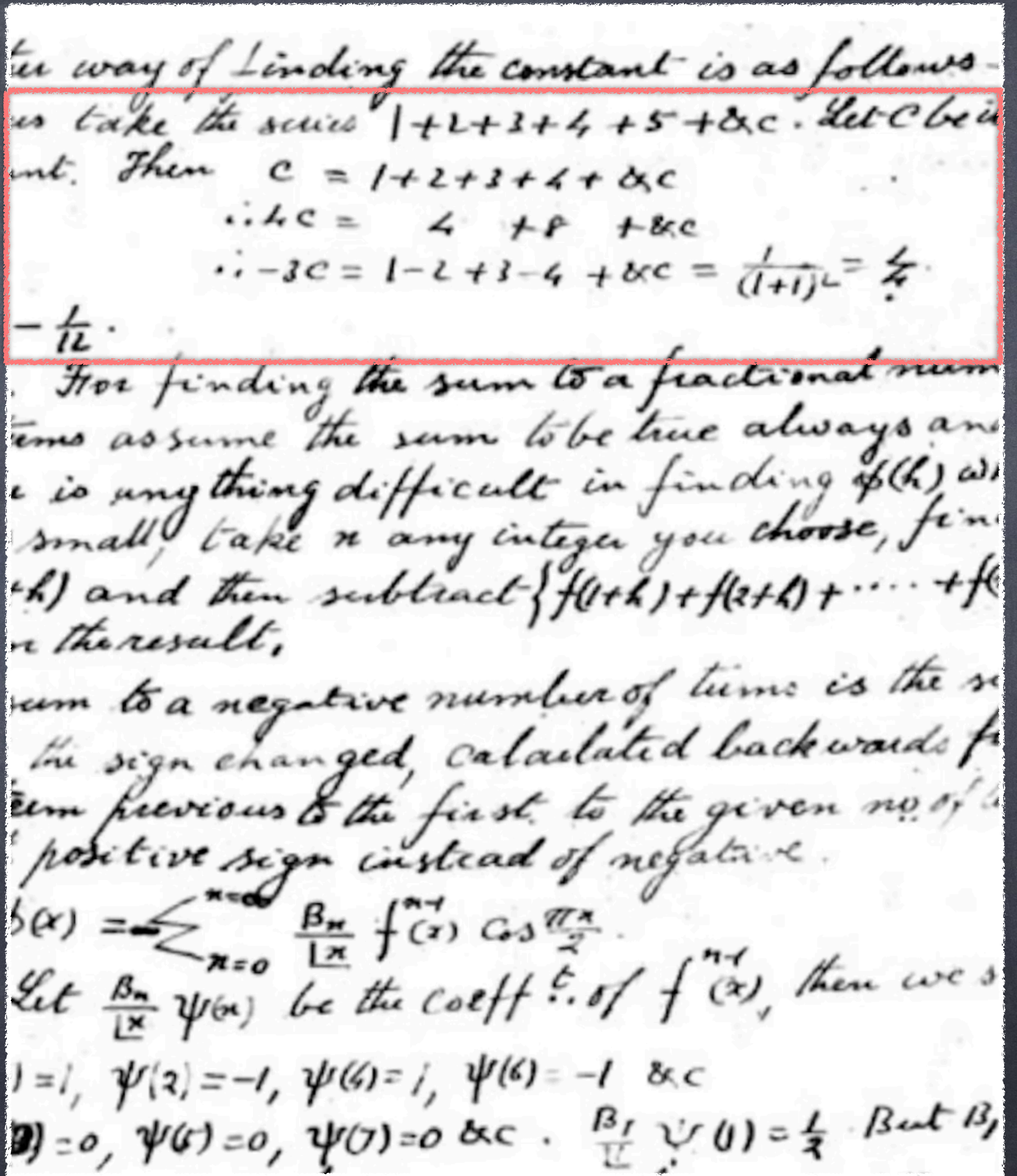
Hardy and Ramanujan

Divergent series

Letter of Ramanujan addressed to G. H. Hardy containing the Claim

$$1+2+3+\cdots=-\frac{1}{12}$$

We will explain the intriguing proof of Ramanujan from his notes on the right.



...the way of finding the constant is as follows -
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...nt. Then $C = 1+2+3+4+\&c$

$\therefore 4C = \quad 4 \quad +8 \quad +\&c$

$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$

$\therefore C = -\frac{1}{12}$.

For finding the sum to a fractional num...
...terms assume the sum to be true always an...
...is anything difficult in finding $\phi(h)$ w...
...small, take n any integer you choose, fin...
...$+h)$ and then subtract $\{f(1+h)+f(2+h)+\cdots+f($...
...n the result,

...sum to a negative number of terms is the s...
...the sign changed, calculated backwards f...
...term previous to the first to the given no. of ...
positive sign instead of negative.

$\phi(x) = \sum_{n=0}^{n=\infty} \frac{B_n}{\lfloor n} f^{n-1}(x) \cos\frac{\pi x}{2}$.

Let $\frac{B_n}{\lfloor n}\psi(n)$ be the coeff^t of $f^{n-1}(x)$, then we s...

$\psi(1)=1$, $\psi(2)=-1$, $\psi(4)=1$, $\psi(6)=-1$ &c

$\psi(3)=0$, $\psi(5)=0$, $\psi(7)=0$ &c. $\frac{B_1}{\lfloor 1}\psi(1)=\frac{1}{2}$ But B_1...

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- Hence $C = -\frac{1}{12}$

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- Well, if the infinite sum $1 + 2 + 3 + \cdots$ is not known to be convergent, then to say that $C - 4C = -3C$ is not legitimate. One of the issues is that of subtracting infinities.
- Moreover, it is not clear, unless the infinite sum $1 - 2 + 3 - 4 + \cdots$ is convergent, why it should equal $\frac{1}{4}$.

$$1 - 2 + 3 - 4 + \dots \stackrel{?}{=} \frac{1}{4}$$

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- Recall that
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- $1 + 2x + 3x^2 + \cdots = \frac{1}{(1+x)^2}.$
- Now, evaluating at $x = -1$ (which is not legitimate), we get the desired sum.

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- However, the sequence of partial sums is $1,0,1,0,\ldots$ and it doesn't have a limit.
- Clearly, violating the requirement $|x| < 1$ leads to meaningless consequences (put $x = 2$).

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- How do we make sense of these sums?

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- We notice that the elements in the sequence of the partial sums are at most 2 and they keep getting closer to 2.
- Moreover, this sequence has the limit 2.

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- This need not be the only way that one might be able to make sense of an infinite series. Are there are other ways?

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- Then it makes sense to multiply such a series by a scalar α and obtain a new series $\alpha a_1 + \alpha a_2 + \alpha a_3 + \cdots$, which is convergent and the sum is αA. Also, $a_1 + a_2 + \cdots + b_1 + b_2 + \cdots = A + B$.

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- Caution: All this is valid only if the initial series is (somehow) known to be convergent.

Multiplying infinite series

- The product of two infinite series is the series $c_1 + c_2 + c_3 + \cdots$, where $c_k = \sum_{i=1}^{k} a_{k-i} b_i$.
- The product of two convergent series need not be convergent.
- An example: $\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots\right)^2$.

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- This new sequence of averages converges!

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- We have a little more, namely, the product of two Cesaro stumble infinite series is also Cesaro summable.
- An convergent series is Cesaro summable.

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- To justify this, recall that the earlier proof involved only multiplying the original convergent series R with the scalar r and the subtraction $R - rR$

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What about $1-2+3-4+\cdots$?

- The sequence of partial sums is $1,-1,2,-2,\ldots$ The average of this sequence is $1,0,\frac{2}{3},0,\frac{3}{5},0,\frac{4}{7}\ldots$
- Well, this is not a convergent sequence either. So, what do we do.
- Let us take the average of the averages and get $1,\frac{1}{2},\frac{5}{9},\frac{5}{12},\frac{34}{75},\frac{34}{90},\ldots$

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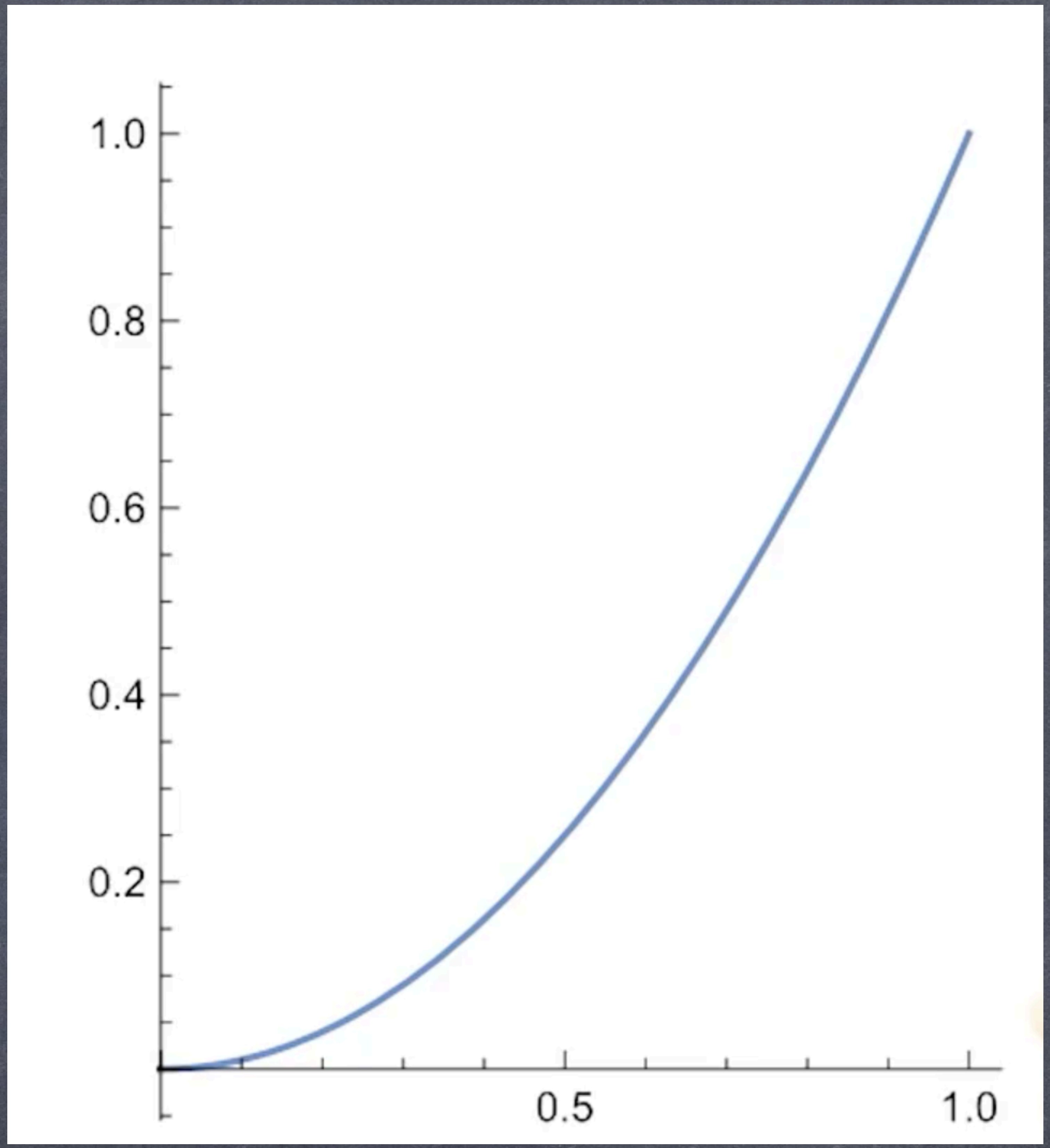
- Can we ever make sense of $1 + 2 + 3 + 4 + \cdots$

The graph of the smooth function x^2

on the right has several (infinitely many) smooth but distinct extensions.



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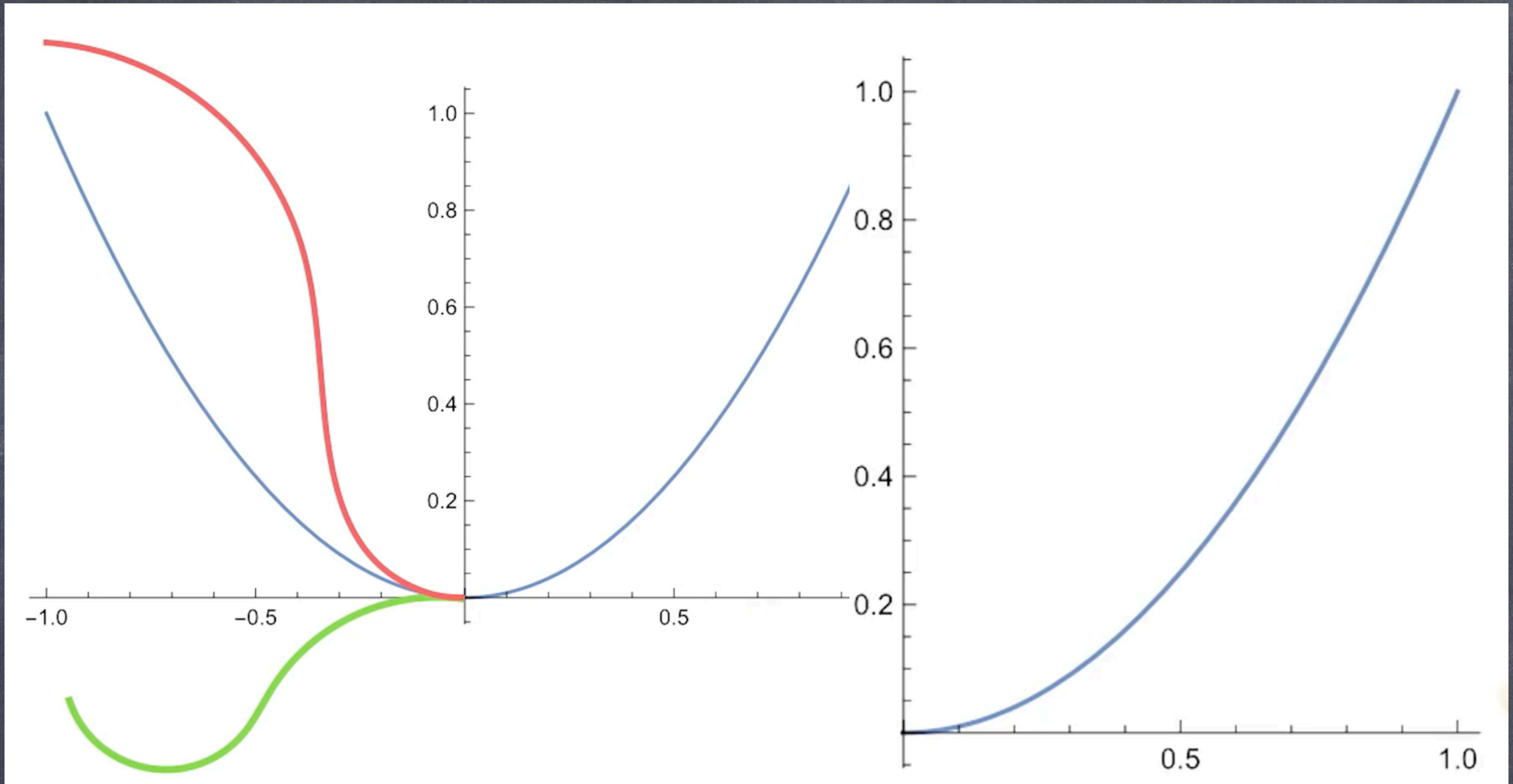

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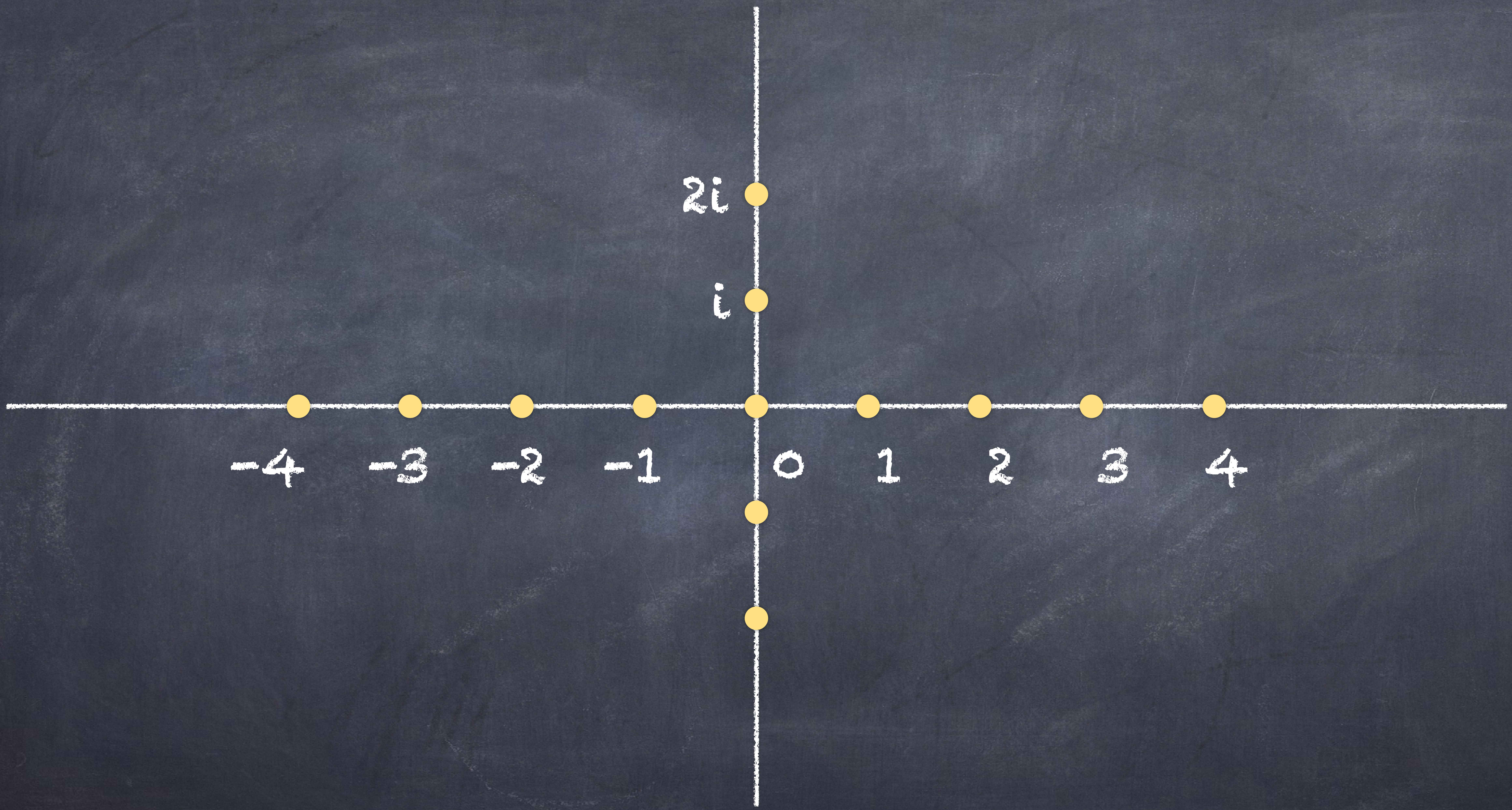
2i
i
-4
-3
-2
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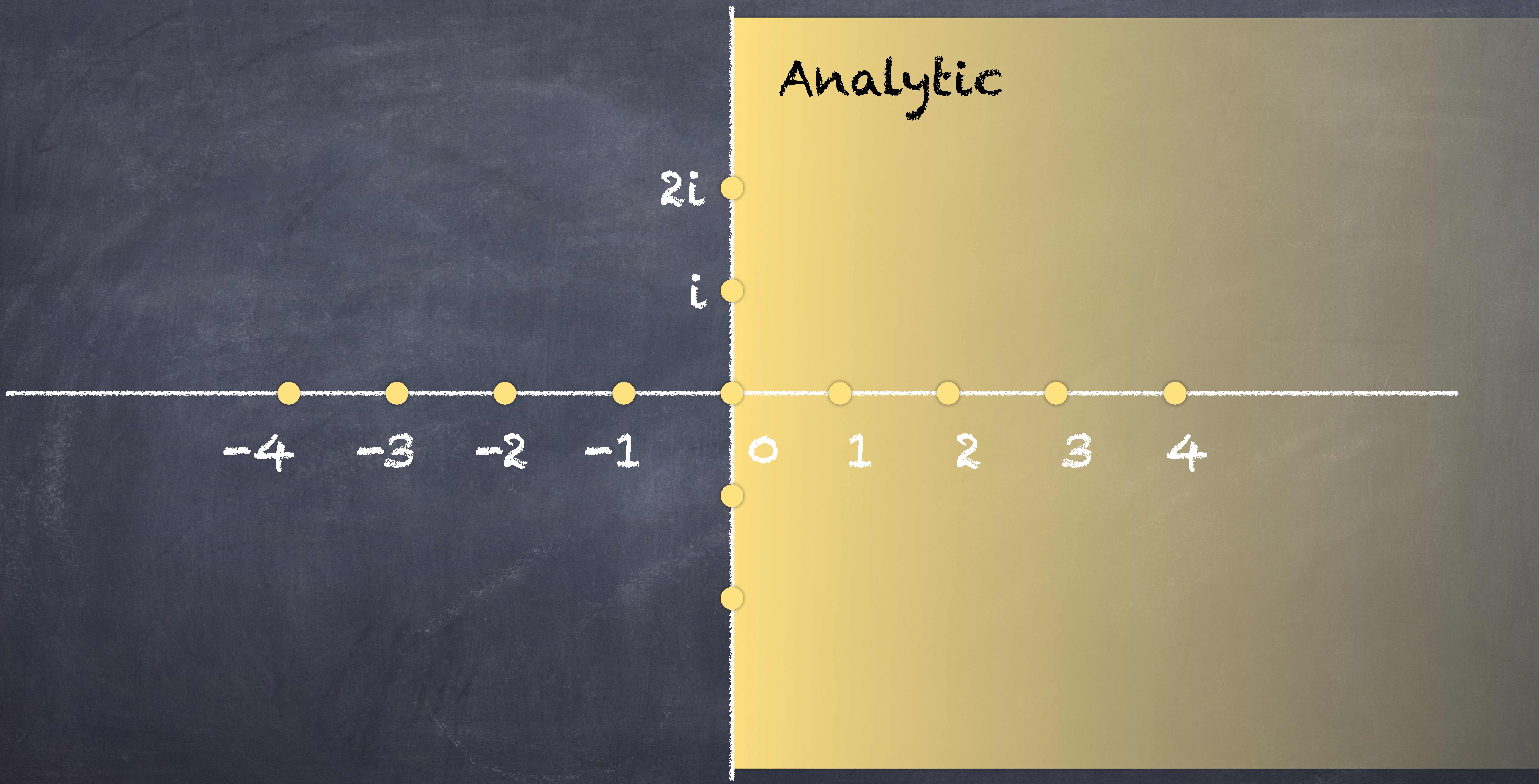
0
1
2
3
4


Analytic
2i
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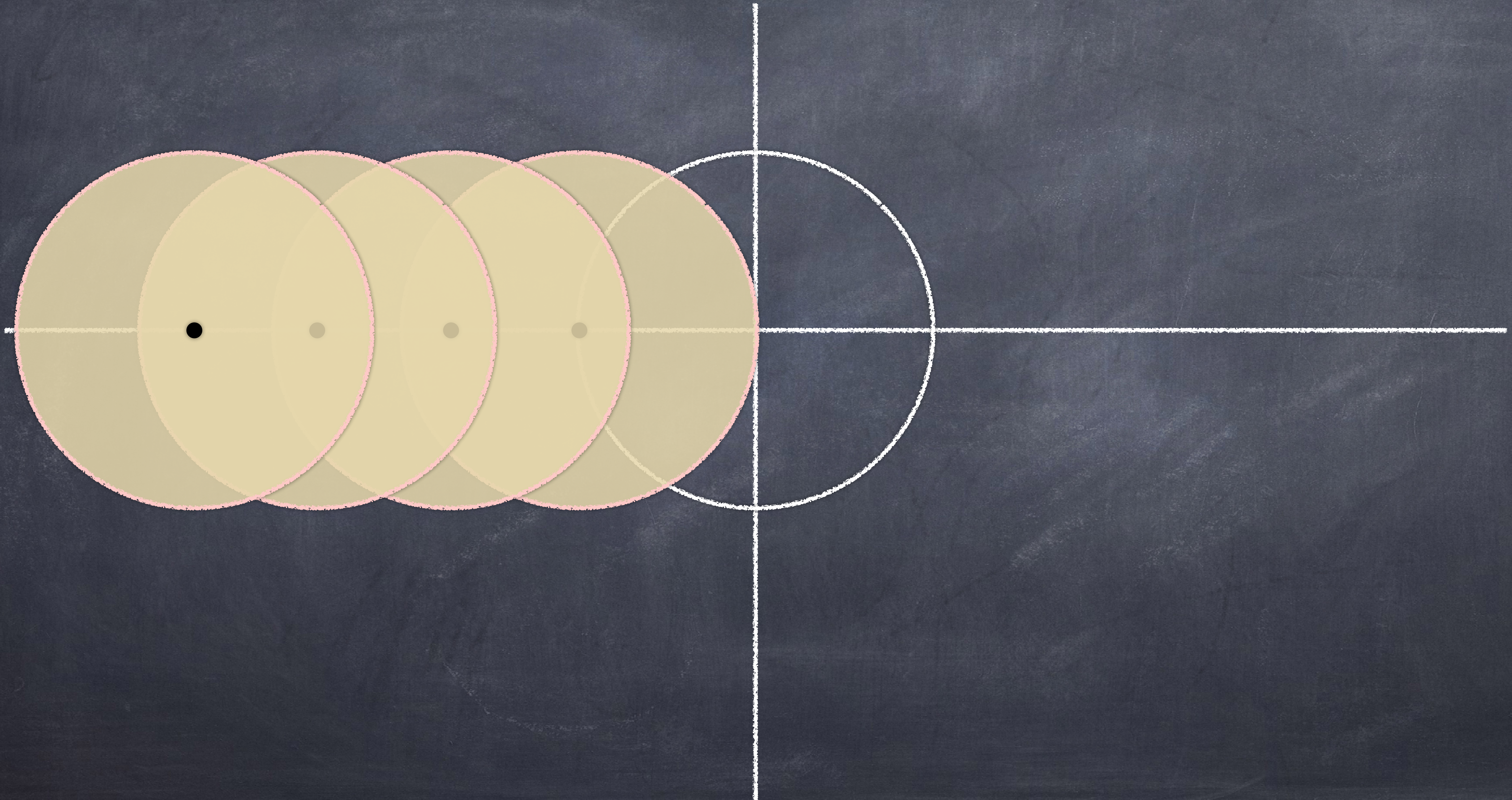
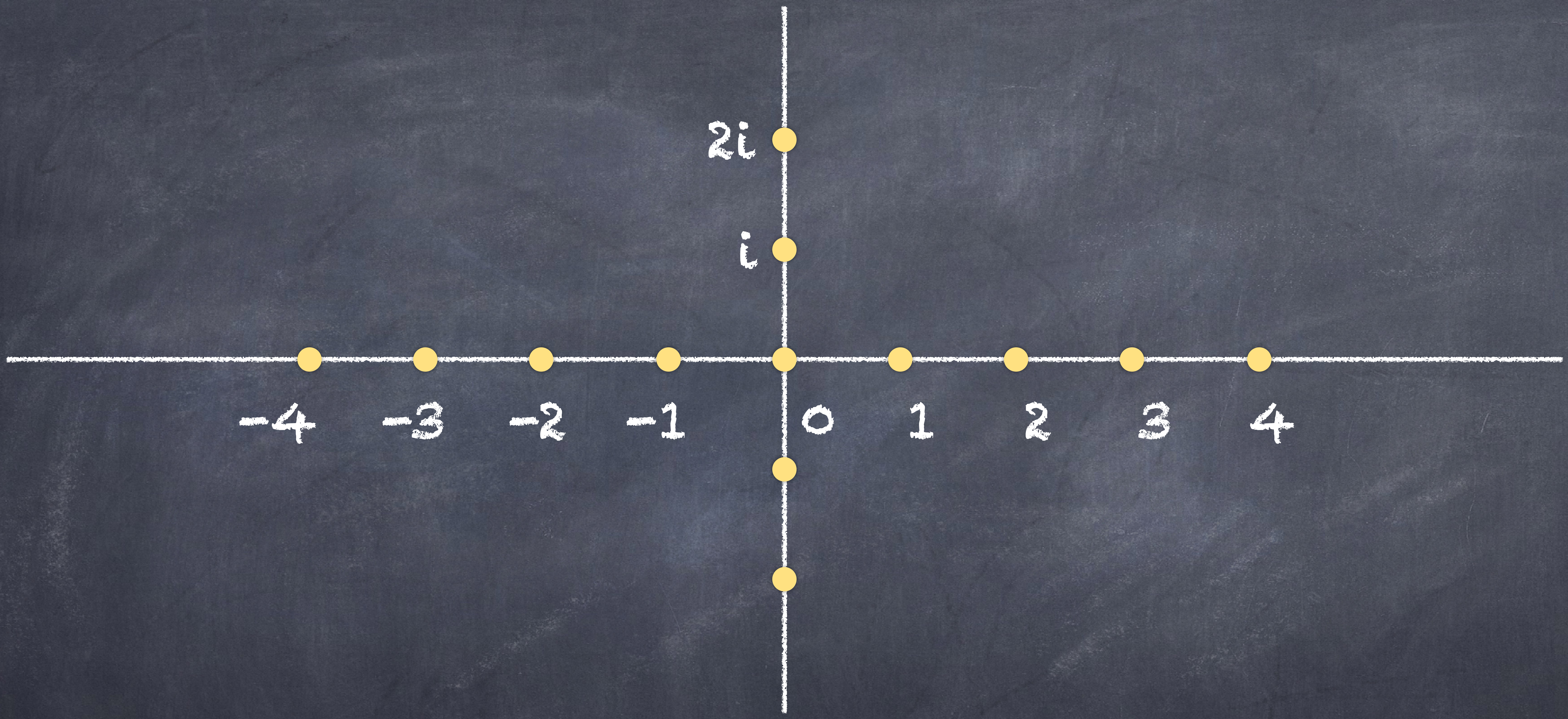
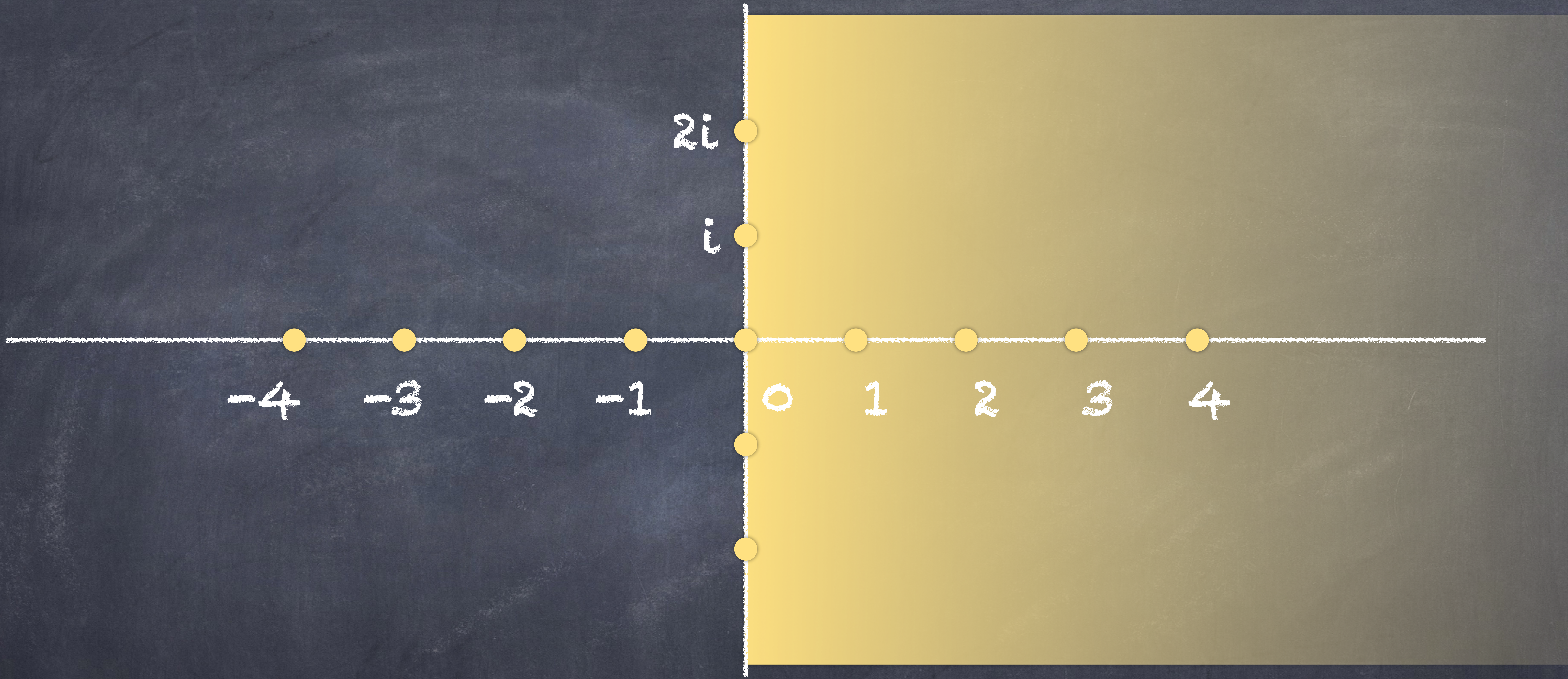
- If we are given an analytic function defined on the right half plane, how many different ways, can we extend to the left?
- Unlike the case of smooth functions, if there is such an extension, then it is unique.

Analytic Continuation





$$f(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots$$



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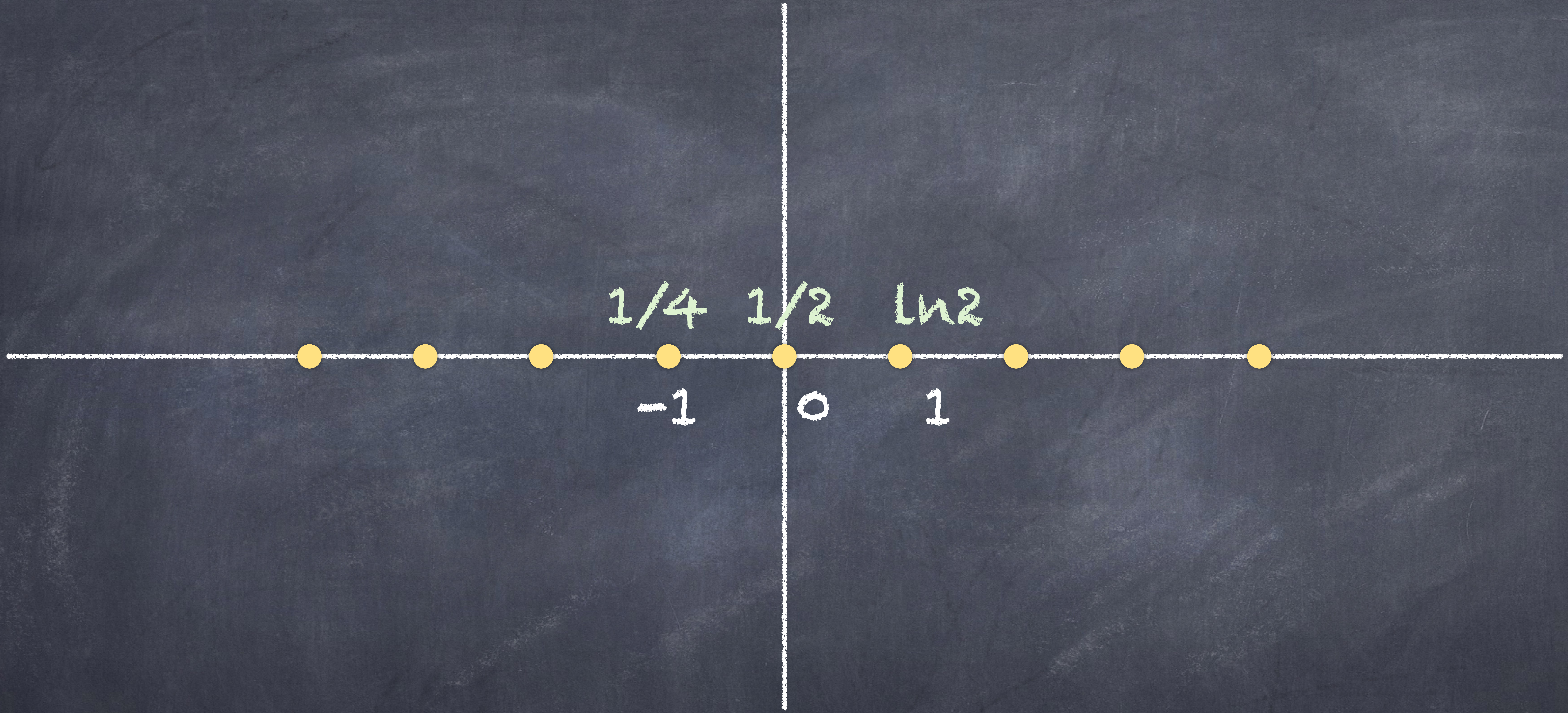
- The series $f(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots$ converges on the right half plane.



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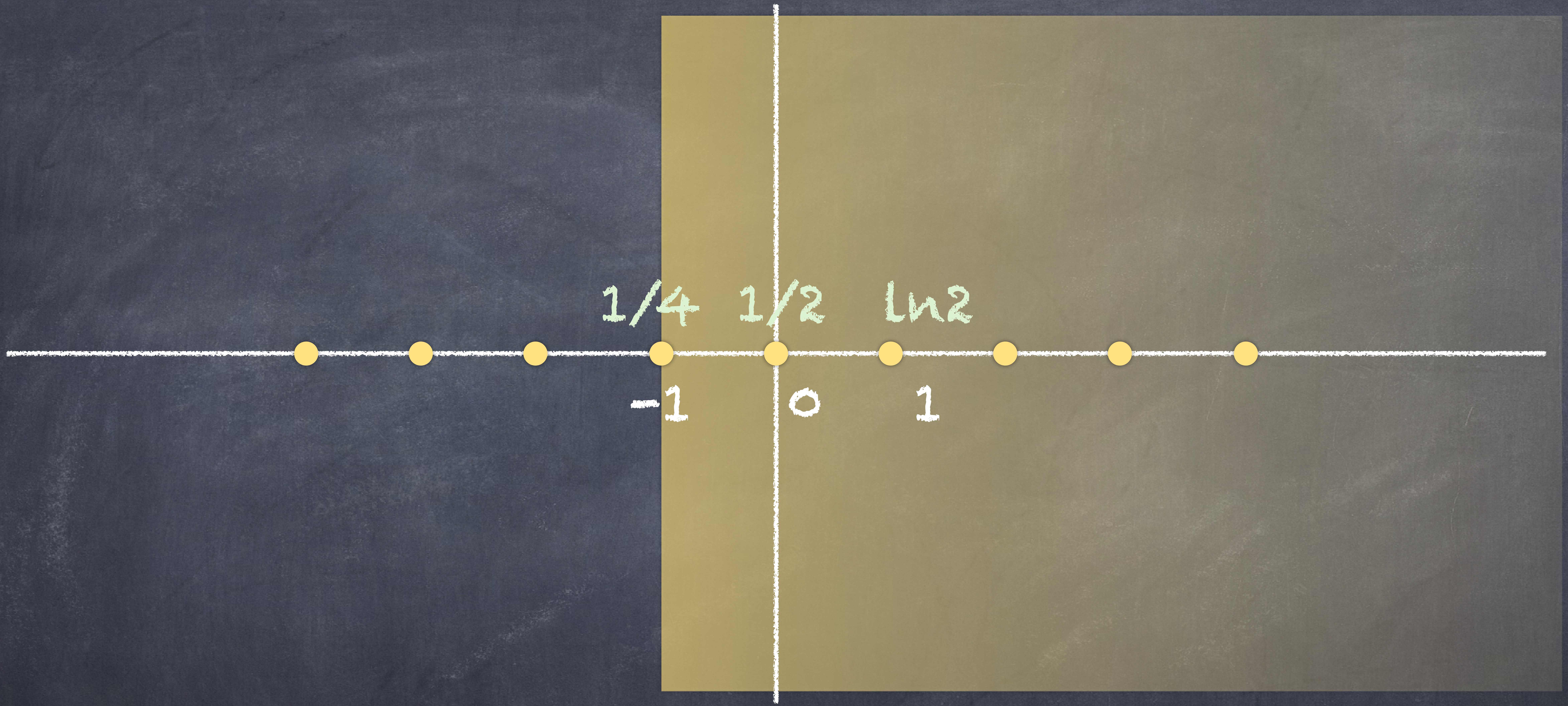

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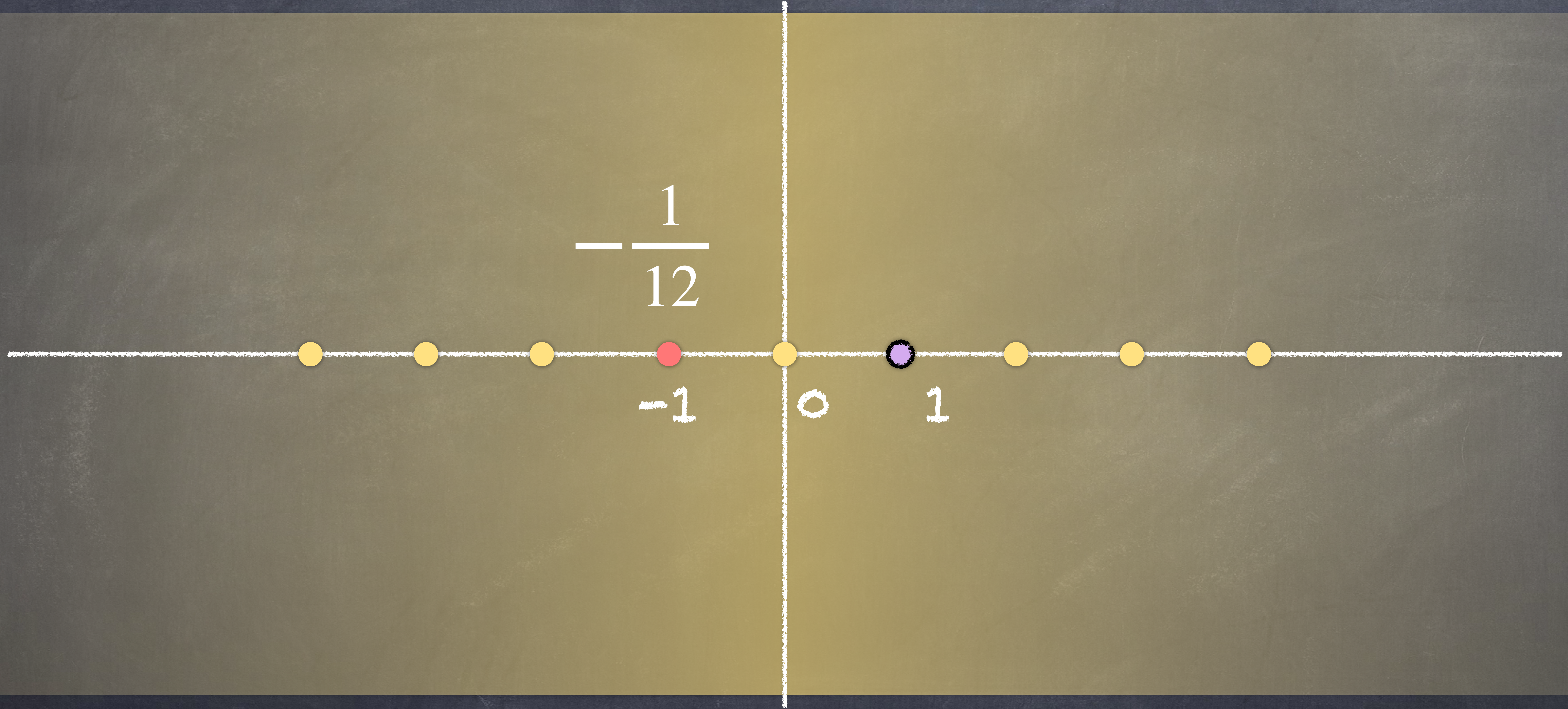
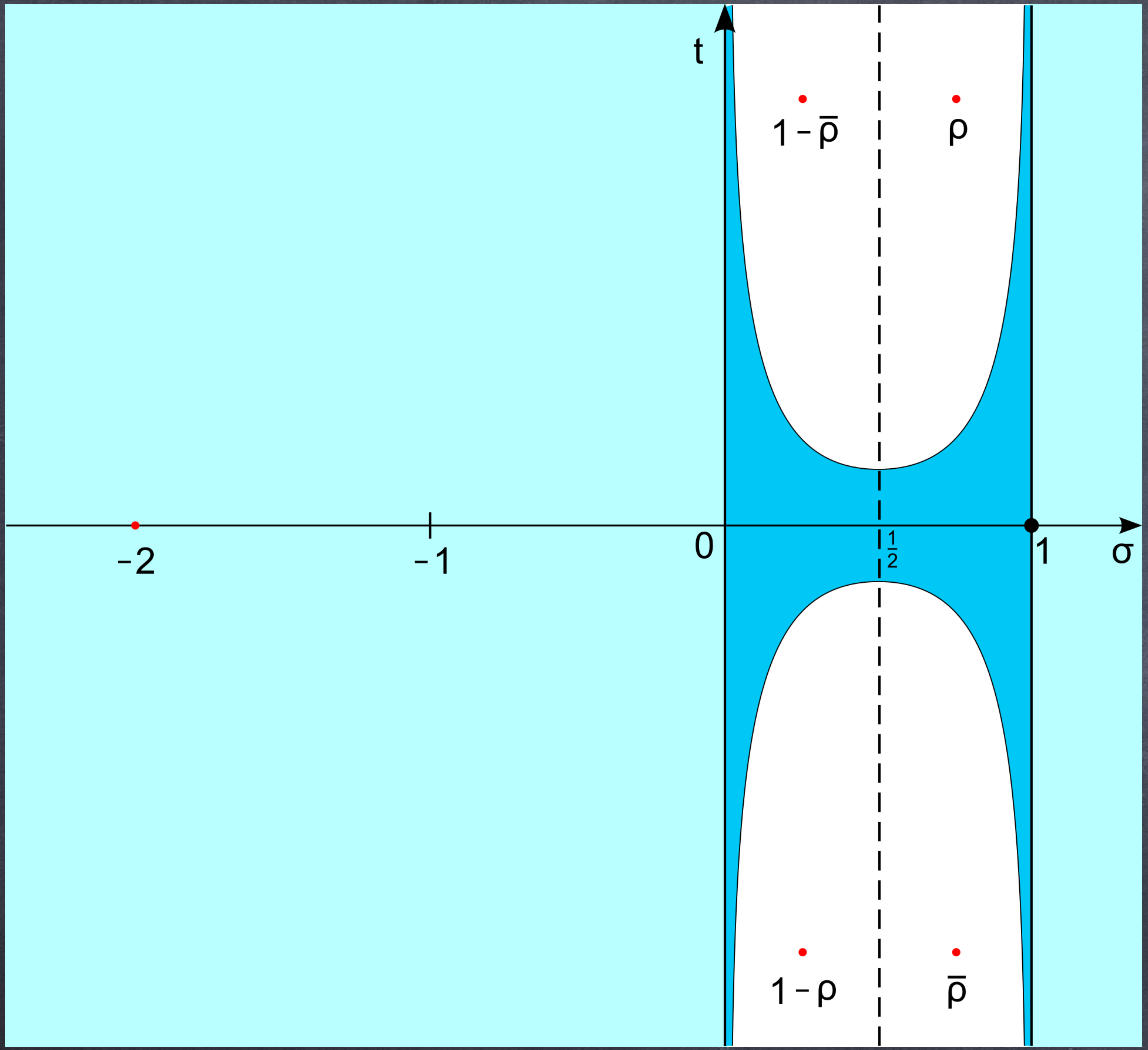
- It converges, when the σ of $z = \sigma + it$ is < 1.
- It does not converge if $\sigma > 1$. But it has an analytic continuation to the entire complex plane except

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots$$



$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$



Apart from the trivial zeros, the Riemann zeta function has no zeros to the right of $\sigma = 1$ and to the left of $\sigma = 0$ (neither can the zeros lie too close to those lines). Furthermore, the non-trivial zeros are symmetric about the real axis and the line $\sigma = \frac{1}{2}$ and, according to the Riemann hypothesis, they all lie on the line $\sigma = \frac{1}{2}$.

THANK YOU!